Combinatorics

As we mentioned last class, the ideas presented in "counting" are core to probability. Counting is like the foundation of a house (where the house is all the great things we will do later in CS109, such as machine learning). Houses are awesome. Foundations on the other hand are pretty much just concrete in a hole. But don't make a house without a foundation. Trust me on that.

Permutations

> **Permutation Rule**: A permutation is an ordered arrangement of *n* distinct object. Those *n* objects can be permuted in n x (n – 1) x (n – 2) x ... x 2 x 1 = n! ways.

This changes slightly if you are permuting a subset of distinct objects, or if some of your objects are indistinct. We will handle those cases shortly!

Example 1

Part A: iPhones have 4 digit passcode. What if there are 4 smudges over 4 digits on screen. How distinct many passcodes possible?

Solution: Since the order of codes is important we should use permutations. And since there are exactly four smudges we know that each number is distinct. Thus we can plug in the permutation formula: 4! = 24

Part B: What if there are 3 smudges over 3 digits on screen?

Solution: One of 3 digits is repeated, but don't know which one. Solve this by making three cases (each with the same number of permutations). Let A, B, C represent 3 digits:
4! permutations of: A B C_1 C_2
But need to eliminate over counting of permutations of C's
3 x [4! / (2! 1! 1!)] = 3 x 12 = 36

Part C: What if there are 2 smudges over 2 digits on screen?

Solution: There are two possibilities, 2 digits used twice each or 1 digit of 2 digits used 3 times, and other digit used once.
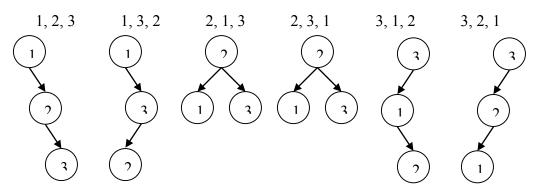[4! / (2! 2!)] + 2 x [4! / (3! 1!)] = 6 + (2 x 4) = 6 + 8 = 14

Example 2
Recall the definition of a **binary search tree** (BST), which is a binary tree that satisfies the following three properties for *every* node n in the tree:

1. n's value is greater than all the values in its left subtree.
2. n's value is less than all the values in its right subtree.
3. both n's left and right subtrees are binary search trees.

Problem: How many possible BST containing 1, 2, and 3 have a degenerate structure (i.e., each node in the BST has at most one child)?

Solution: There are 3! ways to order elements 1, 2, and 3 for insertion:

We see that there are 4 degenerate BSTs here (the first two and last two).

Permutations of Indistinct Objects

Permutation of Indistinct Objects: Generally when there are n objects and
   n_1 are the same (indistinguishable) and
   n_2 are the same and
   ...
   n_r are the same, then there are $\dfrac{n!}{n_1!n_2!...n_r!}$ permutations

Example 3

Problem: How many distinct bit strings can be formed from three 0's and two 1's?

Solution: 5 total digits = 5!

But that is assuming the 0's and 1's are indistinguishable (to make that explicit let's give each one a subscript). Here is a subset of the permutations.

$$0_1 \ 1_1 \ 1_2 \ 0_2 \ 0_3$$
$$0_1 \ 1_1 \ 1_2 \ 0_3 \ 0_2$$
$$0_2 \ 1_1 \ 1_2 \ 0_1 \ 0_3$$
$$0_2 \ 1_1 \ 1_2 \ 0_3 \ 0_1$$
$$0_3 \ 1_1 \ 1_2 \ 0_1 \ 0_2$$
$$0_3 \ 1_1 \ 1_2 \ 0_2 \ 0_1$$

All listed permutations are the same. For any given permutation, there are 3! ways of rearranging the 0s and 2! ways of rearranging the 1s (resulting in an indistinguishable string). We have over counted. Using the permutations of indistinct objects we correct for the over counting:

$$\text{Total} = \frac{5!}{3!2!} = \frac{120}{6 \cdot 2} = \frac{120}{12} = 10$$

Combinations

> **Combinations**: A combination is an unordered selection of r objects from a set of n objects. If all objects are distinct, then the number of ways of making the selection is:
>
> $$\frac{n!}{r!(n-r)!} = \binom{n}{r} \text{ ways}$$
>
> This is often stated as "n choose r"

Consider this general way to product combinations: To select r unordered objects from a set of n objects, E.g. "7 choose 3",

1. First consider permutations of all n objects. There are $n!$ ways to do that.
2. Then select the first r in the permutation. There is one way to do that.
3. Note that the order of r selected objects is irrelevant. There are $r!$ ways to permute them. The selection remains unchanged.
4. Note that the order of $(n-r)$ unselected objects is irrelevant. There are $(n-r)!$ ways to permute them. The selection remains unchanged.

$$\textbf{Total} = \frac{n!}{r!(n-r)!} = \binom{n}{r} = \binom{n}{n-r} \qquad \frac{7!}{3!4!} = 35$$

Which is the combinations formula.

Example 4

Problem: In the Hunger Games, how many ways are there of choosing 2 villages from district 12, which has a population of 8,000?

Solution: This is a straightforward combinations problem. 8,000 choose 2 = 31,996,000.

Example 5

Part A: How many ways to select 3 books from 6

Solution: If each of the books are distinct then this is another straightforward combination problem. There are 6 choose 3 ways:

$$\text{Total} = \binom{6}{3} = \frac{6!}{3!3!} = 20$$

Part B: How many ways to select 3 books if there are two books that should not both be chosen together (for example, don't chose both the 8th and 9th edition of the Ross textbook).

Solution: This problem is easier to solve if we split it up into cases. Consider the following three different cases:

Case 1: Select 8the Ed and 2 other non-9th Ed: There are 4 choose 2 ways of doing so.

Case 2: Select 9th Ed and 2 other non-8th Ed: There are 4 choose 2 ways of doing so.

Case 3: Select 3 from the books that are neither the eighth nor the ninth edition: There are 4 choose 3 ways of doing so.

Using our old friend the Sum Rule of Counting, we can add the cases

$$\text{Total} = 2 * \binom{4}{2} + \binom{4}{3} = 16$$

Alternatively, we could have calculated all the ways of selecting 3 books from 4, and then subtract the "forbidden" ones (eg the selections that break the constraint). Chris calls this the Beijing method because of the Forbidden City there. That's not important

Forbidden Case: Select 8th edition and 9th edition and 1 other book. There are 4 choose 1 ways of doing so (which equals 4).

Answer = All possibilities – forbidden =20 – 4 = 16

Two different ways to get the same right answer!

Group Assignment

You have probably heard about the dreaded "balls and urns" probability examples. What are those all about? They are the many different ways that we can think of stuffing elements into containers. Why people called their containers urns, ~~I have no idea~~ (I looked it up. It turns out that Jacob Bernoulli was into voting and ancient Rome. And in ancient Rome they used urns for ballot boxes). Group assignment problems are useful metaphors for many counting problems.

Example 6
Problem: Say you want to put n distinguishable balls into r urns. (no wait don't say that). Ok fine. No urns. Say we are going to put n strings into r buckets of a hashtable where all outcomes are equally likely. How many possible ways are there of doing this?
Answer: You can think of this as n independent experiments each with r outcomes. Using our friend the General Rule of Counting this comes out to r^n

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Divider Method: A divider problem is one where you want to place *n* indistinguishable items into *r* containers. The divider method works by imagining that you are going to solve this problem by sorting two types of objects, your *n* original elements and *(r – 1)* dividers. Thus you are permuting $n + r - 1$ objects, *n* of which are same (your elements) and *r – 1* of which are same (the dividers). Thus:

$$\text{Total ways} = \frac{(n+r-1)!}{n!(r-1)!} = \binom{n+r-1}{r-1}$$

Example 7

Part A: Say you are an investor at a micro loan group (say the Gramin Bank) and you have $10 thousand to invest in 4 companies (in 1K increments). How many ways can you allocate it?

Solution: This is just like putting 10 balls into 4 urns. Using the Divider Method we get:

$$\text{Total ways} = \binom{10+4-1}{4-1} = \binom{13}{3} = 286$$

Part B: What if you don't have to invest all 10 K? **(Economy tight)**

Solution: Simply imagine that you have an extra company – yourself. Now you are investing 10 thousand in 5 companies. Thus the answer is the same as putting 10 balls into 5 urns.

$$\text{Total ways} = \binom{10+5-1}{5-1} = \binom{14}{4} = 1001$$

Part C: Want to invest at least 3 thousand in company 1?

Solution: There is one way to give 3 thousand to company 1. The number of ways of investing the remaining money is the same as putting 7 balls into 4 urns.

$$\text{Total ways} = \binom{7+4-1}{4-1} = \binom{10}{3} = 120$$

This handout was made fresh just for you. Did you notice any mistakes? Let Chris know and he will fix them.